JHEP HYPER VERSION

Received: *August 17, 2001*, Accepted: *October 5, 2001*
Revised : *Octber 4, 2001*

Poisson Chern-Simons gauge theory

Bogdan Morariu

Department of Physics, Rockefeller University
New York, NY 10021, USA
E-mail: morariu@summit.rockefeller.edu

Abstract: We formulate Poisson Chern-Simons gauge theories on compact group manifolds. These describe a sector of the large representation limit of noncommutative Chern-Simons in the same way as the light-cone formulation of the membrane action describe a sector of the large N matrix model. While the formulation we give is on a group manifold, only excitations that are invariant under the left action of the stability group of a weight are allowed.

Keywords: Chern-Simons Theories, D-branes, M(atrix) Theories, Non-Commutative Geometry.

JHEP10(2001)010

Contents

JHEP10(2001)010

1. Introduction

The study of noncommutative gauge theories has been a popular subject since their relation to matrix theory [1] and String theory [2]–[5] has been understood. They are interesting because they preserve some of the nonlocal properties inherent in String theory. For example, T-duality is a manifest symmetry [6, 7]. For a recent review of noncommutative gauge theory see [8].

It is generally assumed that in the limit of a vanishing noncommutativity parameter a noncommutative gauge theory reduces to a commutative one. Whether this is the case depends on the precise way the limit is taken. The infinitesimal gauge transformations in a noncommutative gauge theory take the form

$$\delta\hat{A} = d\hat{\epsilon} - i[\hat{A} \overset{\star}{,} \hat{\epsilon}]\,.$$

Let us introduce the rescaled quantities $A = \theta\hat{A}$ and $\epsilon = \theta\hat{\epsilon}$, where θ is the noncommutative parameter. Then, gauge transformations written in terms of the rescaled quantities take the form

$$\delta A = d\epsilon + \frac{1}{i\theta}[A \overset{\star}{,} \epsilon]\,.$$

If we take the $\theta \to 0$ limit the second term does not vanish, rather it becomes a Poisson bracket

$$\delta A = d\epsilon + \{A, \epsilon\}\,. \tag{1.1}$$

I will refer to these type of gauge transformation as a Poisson gauge transformation. Alternatively (1.1) represents a gauge transformation of the area preserving diffeomorphism group. It has appeared in the light cone formulation of the membrane action [9] and in the lagrangian fluid dynamics formulation of the quantum Hall effect[1] [10]. It was also discussed in the context of noncommutative gauge theories of D-branes [11].

Here we will present another example, the Poisson Chern-Simons gauge theory on a group manifold. It can be obtained formally as the large representation limit of the noncommutative Chern-Simons gauge theory discussed in [12, 13, 14]. However, I will introduce it independently of its noncommutative partner. One reason to study these models is that we can understand their geometry. This is not always possible for the noncommutative versions. In particular, the "dimension" of a noncommutative space is not a very well defined quantity; it only makes sense in the commutative limit. In the commutative limit for the SU(2) case, we can clarify the relation between the fuzzy sphere of [12, 13] and the the three-fold of [14]. In particular we find that while the action can be naturally written on a three dimensional manifold, the component fields must be invariant under a U(1) action so they naturally live on a two sphere.

The plan of the paper is as follows. In section 2, I will review the formulation of gauge theory on a group manifold. In section 3, I describe how to introduce a Poisson bracket on functions on the group which are invariant under the left action of a stability subgroup. This is closely related to the Lie-Poisson bracket on the dual of the Lie algebra of the group. I will also extend the bracket to the exterior algebra and use it to give a BRST formulation of the calculus. In the last section I will introduce the Poisson Chern-Simons action, first for the SU(2) case, and then for an arbitrary compact Lie group G. For $G \neq \mathrm{SU}(2)$ I will also point out some difficulties with interpreting the noncommutative Chern-Simons, as proposed in [14], as a toy model for Witten's string field theory [15].

2. Gauge theory on group manifolds

In order to fix the notation and to make the paper more self-contained, I will give a brief review of gauge theory on a group manifold G. I will assume that G is compact, connected and simply-connected. For simplicity I will also assume that G is simple, but most of what follows also works for semi-simple groups. Since the Lie algebra valued 1-form $g^{-1}\mathrm{d}g$ is left invariant we have the following expansion

$$g^{-1}\mathrm{d}g = T_a c^a \,,$$

where the T_a's are antihermitean and satisfy the Lie algebra $[T_a, T_b] = f_{ab}{}^{c}\, T_c$, and c^a are left invariant 1-forms. The forms c^a give a globally defined vielbein on the group manifold. The exterior derivative acting on a function F on the group can be written as

$$\mathrm{d}F = c^a\, L_a(F)\,, \tag{2.1}$$

[1] The spatial coordinate functions of the fluid are defined as $X^i = x^i + \theta\epsilon^{ij}\hat{A}_i$ thus they are related to A_i by a translation and a rotation.

where L_a are left invariant vector fields satisfying $[L_a, L_b] = f_{ab}{}^c L_c$. Explicitly, they are given by

$$L_a = \mathrm{tr}\left(T_a g \frac{\partial}{\partial g^T}\right).$$

We can also extend (2.1) to the exterior derivative on an arbitrary form. Let b_a be an operator acting on a form ω as

$$b_a(\omega) = i_{L_a}\omega\,,$$

where i denotes the inner product of the vector field L_a with the form ω. Then the exterior derivative can be written as

$$\mathrm{d} = c^a L_a - \frac{1}{2} f_{ab}{}^c\, c^a c^b b_c\,, \tag{2.2}$$

where we assume that L_a only acts on the component functions. Note that b_a and c^a are odd and satisfy the ghost anti-commutation relations $\{b_a, c^d\} = \delta_a^d$.

Let us now consider gauge theory on the group manifold G. Later we will mainly be interested in a U(1) gauge theory but for now we will write our formulae so that they can also be used in the nonabelian case.[2] The gauge potential A can be written globally as

$$A = c^a A_a\,, \tag{2.3}$$

where we have made an implicit choice of the bundle. Using the definition of the exterior derivative (2.2), one can show that the components of the field strength $F = \mathrm{d}A - iA^2 = \frac{1}{2}c^a c^b F_{ab}$ take the form

$$F_{ab} = L_a(A_b) - L_b(A_a) - i[A_a, A_b] - f_{ab}{}^c A_c\,.$$

Similarly, one can show that the global and infinitesimal gauge transformations have the form

$$\begin{aligned} A'_a &= g^{-1} A_a g + i g^{-1} L_a(g)\,, \\ \delta A_a &= L_a(\epsilon) - i[A_a, \epsilon]\,. \end{aligned} \tag{2.4}$$

The Yang-Mills action is given by

$$\mathcal{S}_{YM} = \int \mathrm{d}g\, \frac{1}{4}\mathrm{tr}\,(F_{ab}F^{ab})\,,$$

where the tangent space indexes are raised and lowered with δ_{ab} and $\mathrm{d}g$ denotes the Haar measure on the group.

[2] One should not confuse the commutators in this section which would not be present in an abelian theory, with the $\star$-commutators appearing in a noncommutative gauge theory.

The Chern-Simons form $\omega_{CS} = \mathrm{tr}(A\mathrm{d}A + \frac{2}{3i}A^3)$ takes the following form

$$\omega_{CS} = c^a c^b c^c \mathrm{tr}\left(A_a L_b(A_c) - \frac{1}{2}A_a f_{bc}{}^d A_d + \frac{1}{3i}A_a[A_b, A_c]\right).$$

For the three-dimensional group SU(2) , we can also define a Chern-Simons action [16]

$$\mathcal{S}_{CS} = \frac{k}{4\pi}\int \omega_{CS}\,,$$

by integrating the 3-form ω_{CS} over the group manifold.

3. Poisson algebra and BRST calculus

For a U(1) gauge theory the gauge transformation (2.4) is simply $\delta A_a = L_a(\epsilon)$. We would like to modify the U(1) Yang-Mills and Chern-Simons actions above in such a way that the gauge transformation becomes similar to the one discussed in the introduction

$$\delta A_a = L_a(\epsilon) + \{A_a, \epsilon\}\,. \tag{3.1}$$

First, however we must pick an appropriate Poisson bracket. While there are known examples of Poisson brackets on groups (related to quantum groups) we will not be interested in these. Instead I will consider another bracket which can only be defined on a subset of functions on the group. Let λ be an element of the Lie algebra of G. We assume that G is semi-simple so we can also think of λ as an element of the dual of the Lie algebra. We denote by H_λ the stability subgroup of λ i.e. $[H_\lambda, \lambda] = 0$. Then the bracket can only be used on functions F invariant under left multiplication by elements h of H_λ

$$F(hg) = F(g)\,. \tag{3.2}$$

These functions can be identified with functions on $H_\lambda\backslash G$ and this space can in turn be identified with the coadjoint orbit on the dual of the Lie algebra of G passing through λ. But this is a symplectic leaf of the Lie-Poisson bracket and thus is has a natural Poisson bracket [19]. If x_a denotes linear coordinates on the dual of the Lie algebra of G, the Lie-Poisson bracket on these generators is given by

$$\{x_a, x_b\} = f_{ab}{}^c x_c\,. \tag{3.3}$$

The map $g \to g^{-1}\lambda g$ can be thought of as a map from $H_\lambda\backslash G$ to the dual of the Lie algebra. Note that two points g and g' belonging to the same coset of $H_\lambda\backslash G$ are mapped to the same point. The algebra of function on the group satisfying (3.2) is generated by

$$x_a = \mathrm{tr}(T_a g^{-1}\lambda g)\,,$$

so in principle all we need is the bracket (3.3). However it is interesting to have the bracket defined directly on the group. One can show that the bracket takes the form

$$\{F, G\} = \mathcal{P}^{ab}\, L_a(F)\, L_b(G)\,, \tag{3.4}$$

where $\mathcal{P}^{ab}$ is defined as follows. First let $M_{ab} \equiv f_{ab}{}^c x_c$. Note that M is a degenerate antisymmetric matrix. If fact the number of zero eigenvalues coincide with the codimension of the symplectic leaf. Then, $\mathcal{P}^{ab}$ is defined to act as *minus* the inverse of M in the subspace where M in nondegenerate and to act as zero in the orthogonal subspace. For SU(2) we have $\mathcal{P}^{ab} = (x^d x_d)^{-1} f^{abc} x_c$ and the Poisson bracket takes the form

$$\{F, G\} = \frac{1}{(x^d x_d)} f^{abc} x_a\, L_b(F)\, L_c(G)\,.$$

We will not need to know the explicit form of $\mathcal{P}^{ab}$ and will use only the fact that it must satisfy

$$M_{ab}\mathcal{P}^{bc}M_{cd} = -M_{ad}\,, \tag{3.5}$$

which is a direct consequence of its definition. To show that (3.4) gives the correct Poisson bracket it is enough to check that on the generators x_a it gives (3.3). This can be easily accomplished using (3.5) Note also, that when one of the entries in (3.4) is x_a , we have $\{x_a, F\} = L_a(F)$.

Next we extend (3.4) from left H_λ-invariant function to left H_λ-invariant forms. Such forms can be expanded in components as

$$\chi = \frac{1}{p!}\, c^{a_1} \cdots c^{a_p}\, \chi_{a_1 \cdots a_p}\,,$$

where $\chi_{a_1 \cdots a_p}$ are left H_λ-invariant functions on the group. The graded bracket is uniquely defined by requiring that it acts as a derivation on each of its entries, that c^a have trivial bracket with the components and satisfy

$$\{c^a, c^b\} = 0\,,$$

and that on components it reduces to the Poisson bracket (3.4).

One can further extend the graded bracket to include the odd inner product operators b_a. Then, since $\{x_a, F\} = L_a(F)$ we can rewrite the exterior derivative acting on forms in BRST form

$$d\chi = \{Q, \chi\}\,,$$

where the BRST function is given by

$$Q = c^a x_a - \frac{1}{2} f_{ab}{}^c c^a c^b b_c\,.$$

Note that $\{Q, Q\} = 0$, and this guarantees the nilpotency of the exterior derivative.

JHEP10(2001)010

4. Poisson Chern-Simons on compact Lie groups

Having defined the Poisson bracket (3.4) and its graded extension, we can now try to find a Poisson Chern-Simons action invariant under the Poisson gauge transformation (3.1). We will do so first for $G = \mathrm{SU}(2)$ and then generalize it to an arbitrary group.

4.1 The SU(2) case

Experience with noncommutative gauge theories has taught us that their actions look similar to nonabelian gauge theories with $\star$-commutators instead of nonabelian commutators. Since these would become Poisson brackets in the limit that we are considering, it is not difficult to guess the form of the Poisson Chern-Simons action

$$\mathcal{S}_{PCS} = \frac{k}{4\pi} \int A\{Q, A\} + \frac{1}{3} A\{A, A\} \,. \tag{4.1}$$

The integral in (4.1) is just the integral of a 3-form on the SU(2) group manifold. It has two important properties. If χ^p denotes a p-form the integral satisfies

$$\int \{Q, \chi^2\} = 0 \,, \tag{4.2}$$

$$\int \{\chi^1, \chi^2\} = \int \{\chi^2, \chi^1\} = 0 \,. \tag{4.3}$$

Since $\{Q, \chi^2\} = \mathrm{d}\chi^2$ the relation (4.2) reflects the fact that SU(2) is a manifold without boundary. I will now sketch how to prove (4.3). First, by writing the forms in components, one can show that (4.3) is a consequence of the following relation

$$\int \mathrm{d}g \, \{F, G\} = 0 \,, \tag{4.4}$$

where F and G are arbitrary left H_λ-invariant functions in the group. We have

$$\begin{aligned} \int \mathrm{d}g \, \{F, G\} &= \int \mathrm{d}g \, \{F, x_a\} \partial_a G \\ &= \int \mathrm{d}g \, [\{F \partial_a G, x_a\} - F\{\partial_a G, x_a\}] \\ &= \int \mathrm{d}g \, [-L_a(F \partial_a G) + F \partial_a \partial_b G \{x_a, x_b\}] = 0 \,. \end{aligned}$$

The last equality follows from the invariance of the Haar measure $\int \mathrm{d}g \, L_a(F) = 0$ and the antisymmetry of the Poisson bracket. Note that (4.4) is true for any compact Lie group not just for SU(2). Using (4.2) and (4.3) it is not difficult to prove the gauge invariance of the action (the Poisson Chern-Simons form $\omega_{PCS} = A\{Q, A\} + \frac{1}{3} A\{A, A\}$ is not invariant under this gauge transformation, only the action is).

The equations of motion derived from (4.1) can be written compactly

$$\{Q+A, Q+A\} = 0\,. \tag{4.5}$$

Note also that the gauge transformation (3.1) can be written as

$$\delta A = \{Q+A, \epsilon\}\,. \tag{4.6}$$

To obtain the spectrum, it is enough to consider the linearized equations of motion (4.5) and gauge transformations (4.6), which take the form

$$\{Q, A\} = 0\,, \qquad \delta A = \{Q, \epsilon\}\,. \tag{4.7}$$

To this order, the solutions of the equations of motion modulo gauge transformations coincide with the cohomology of Q.

In components the Poisson Chern-Simons action takes the form

$$\mathcal{S}_{PCS} = \frac{k}{4\pi}\int \mathrm{d}g\, \epsilon^{abc}\left(A_a\{x_b, A_c\} - \frac{1}{2}A_a f_{bc}{}^{d}A_d + \frac{1}{3}A_a\{A_b, A_c\}\right), \tag{4.8}$$

while the infinitesimal Poisson gauge transformation is given by

$$\delta A_a = L_a(\epsilon) + \{A_a, \epsilon\} = \{x_a + A_a, \epsilon\}\,. \tag{4.9}$$

The equations of motion, written in components, have the form $F_{ab} = 0$, with the field strength F_{ab} given by

$$F_{ab} = L_a(A_b) - L_b(A_a) + \{A_a, A_b\} - f_{ab}{}^{c}A_c = \{x_a + A_a, x_b + A_b\} - f_{ab}{}^{c}(x_c + A_c)\,. \tag{4.10}$$

We have normalized the Poisson Chern-Simons action in a similar way as a standard Chern-Simons action. However the level k does not seem to be quantized.[3] In the noncommutative Chern-Simons theory [12, 13, 14] one can explicitly check the invariance of the action under global gauge transformations. It should also be possible to do this directly in the Poisson Chern-Simons case, but we will not do this here. Even if we generalize the theory in the obvious way to a U(n) Poisson Chern-Simons the level is not quantized. However, note that the lack of quantization of the level is not so surprising. Indeed, if one scales out the Poisson bracket, the action reduces to a standard U(n) Chern-Simons but we still have the invariance under $H_\lambda = U(1)$. Thus instead of $\pi_3(U(n))$, gauge transformations are classified by $\pi_2(U(n))$ which is trivial.

Since all the components of the gauge fields are H_λ-invariant everything could be reformulated as a field theory on a two-sphere. The gauge fields can be locally decomposed into a true two-dimensional gauge field and a two-dimensional scalar

[3]This is in contrast to Chern-Simons on the noncommutative plane where the level is quantized [17, 18].

(representing the component along the H_λ orbit). Thus at the level of fields we are performing a dimensional reduction along the H_λ orbit. Note however that the Poisson bracket is only defined on H_λ-invariant fields and it is inconsistent to use the action (4.1) for arbitrary fields. Furthermore, we are not aware of any other action whose dimensional reduction is (4.1).

4.2 Arbitrary compact Lie groups case

It was noticed in [14] that the noncommutative Chern-Simons gauge theory can be generalized to an arbitrary compact group G. In our case we can arrive at a similar result as follows. In equation (4.8) we can substitute the structure constants of G, (with appropriately placed indices,) for the ϵ-tensor

$$\mathcal{S}_{PCS} \propto \int \mathrm{d}g\, f^{abc} \left(A_a L_b(A_c) - \frac{1}{2} A_a f_{bc}{}^d A_d + \frac{1}{3} A_a \{A_b, A_c\} \right). \tag{4.11}$$

One can check that this action is also invariant under the Poisson gauge transformation (4.9).

We can write the action (4.11) in a more geometric way as follows. On any group manifold we can define the left invariant 3–form

$$\Omega = \mathrm{tr}\,[(g^{-1}\mathrm{d}g)^{\wedge 3}]\,.$$

Then the action can be written

$$\mathcal{S}_{PCS} \propto \int \star\Omega \wedge \omega_{PCS} = \int \mathrm{d}g\, \langle \Omega, \omega_{PCS} \rangle\,, \tag{4.12}$$

where $\star$ denotes the Hodge dual and $\langle\,,\rangle$ denotes the inner product of 3-forms.

However, it is clear that in the action (4.8) the ϵ-tensor comes from the volume of integration and it is an accident (in the three dimensional case) that it coincides with the structure constants of the group SU(2). Thus one wonders if substituting the structure constants for the ϵ-tensor does not have some undesirable consequences. Indeed, while it is true that the action (4.11) is gauge invariant, the equations of motion derived from it read

$$f^{abc} F_{bc} = 0\,, \tag{4.13}$$

and these only imply the vanishing of the field strength if the rank of the group is three. Thus for arbitrary G we do not have $\{Q + A, Q + A\} = 0$. Similarly, in the noncommutative Chern-Simons case the equation $(Q + A)^2 = 0$ claimed in [14] is only valid for $G = \mathrm{SU}(2)$. In Witten's string field theory [15] it is implicitly assumed that the "integral" is nondegenerate, such that the equations of motion are exactly the vanishing of the field strength. Therefore we can only interpret the three dimensional case as a toy model for Witten's string field theory. It is however an interesting problem to find nontrivial solutions of nonvanishing field strength of the equations (4.13).

It is also possible to define a Poisson Yang-Mills action on a group manifold

$$\mathcal{S}_{PYM} = \int \mathrm{d}g \, \frac{1}{4} F_{ab} F^{ab} \, ,$$

where now the field strength is given by (4.10). Note that the lagrangian is only covariant under Poisson gauge transformations and just as in noncommutative gauge theories one has to integrate to obtain a gauge invariant action. Unlike the Poisson Chern-Simons action, the Poisson Yang-Mills action is naturally defined on an arbitrary group G.

We conclude by noting that Poisson gauge theories are interesting because they retain some of the properties of noncommutative gauge theories. For example there are no local Poisson gauge invariant quantities. It would be desirable to develop a better understanding of the observables of these theories. Note also that in a manner similar to the "regularization" of the membrane action discussed in [9], we can quantize the Poisson bracket on the coadjoint orbits [19], (see also [20, 21] for a more recent treatment in the spirit of this paper,) and obtain the noncommutative Chern-Simons action. The quantization leads to unitary representations only if λ is a weight. In this case we obtain the noncommutative Chern-Simons theory corresponding to the irreducible representation whose highest weight is λ. It would be interesting to find the Poisson Chern-Simons theory corresponding to the reducible representations discussed in [14].

Acknowledgments

I would like to thank Klaus Bering and Alexios Polychronakos for useful discussions. This work was supported in part by the U.S. Department of Energy under Contract Number DE-FG02-91ER40651-TASK B.

References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219]; *Constrained quantization of open string in background B field and noncommutative D-brane*, *Nucl. Phys.* **B 568** (2000) 447 [hep-th/9906192].

[3] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Dirac quantization of open strings and noncommutativity in branes*, *Nucl. Phys.* **B 576** (2000) 578 [hep-th/9906161].

[4] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[5] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[6] A. Schwarz, *Morita equivalence and duality*, *Nucl. Phys.* **B 534** (1998) 720 [hep-th/9805034].

[7] D. Brace, B. Morariu and B. Zumino, *Dualities of the matrix model from T-duality of the type II string*, *Nucl. Phys.* **B 545** (1999) 192 [hep-th/9810099];
D. Brace, B. Morariu and B. Zumino, *T-duality and Ramond-Ramond backgrounds in the matrix model*, *Nucl. Phys.* **B 549** (1999) 181 [hep-th/9811213].

[8] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, hep-th/0106048.

[9] J. Hoppe, *Quantum theory of a massless relativistic surface and a two-dimensional bound state problem*, PhD thesis, MIT, 1982;
E. Bergshoeff, E. Sezgin and Y. Tanii, *Hamiltonian formulation of the supermembrane*, *Nucl. Phys.* **B 298** (1988) 187;
B. de Wit, J. Hoppe and H. Nicolai, *On the quantum mechanics of supermembranes*, *Nucl. Phys.* **B 305** (1988) 545.

[10] L. Susskind, *The quantum hall fluid and non-commutative Chern Simons theory*, hep-th/0101029.

[11] L. Cornalba and R. Schiappa, *Matrix theory star products from the Born-Infeld action*, *Adv. Theor. Math. Phys.* **4** (2000) 249 [hep-th/9907211];
L. Cornalba, *D-brane physics and noncommutative Yang-Mills theory*, *Adv. Theor. Math. Phys.* **4** (2000) 271 [hep-th/9909081].

[12] C. Klimcik, *A nonperturbative regularization of the supersymmetric Schwinger model*, *Comm. Math. Phys.* **206** (1999) 567 [hep-th/9903112].

[13] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Brane dynamics in background fluxes and non-commutative geometry*, *J. High Energy Phys.* **05** (2000) 010 [hep-th/0003187].

[14] D.J. Gross and V. Periwal, *String field theory, noncommutative Chern-Simons theory and Lie algebra cohomology*, *J. High Energy Phys.* **08** (2001) 008 [hep-th/0106242].

[15] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[16] E. Witten, *Quantum field theory and the Jones polynomial*, *Comm. Math. Phys.* **121** (1989) 351.

[17] V.P. Nair and A.P. Polychronakos, *On level quantization for the noncommutative Chern-Simons theory*, *Phys. Rev. Lett.* **87** (2001) 030403 [hep-th/0102181].

[18] D. Bak, K. Lee and J.-H. Park, *Chern-Simons theories on noncommutative plane*, *Phys. Rev. Lett.* **87** (2001) 030402 [hep-th/0102188].

[19] B. Kostant, *Lecture notes in mathematics*, vol. 170 Springer 1970;
A. A. Kirillov, *Elements of the theory of representations*, Springer, Berlin, Heidleberg, New York 1976.

[20] G. Alexanian, A.P. Balachandran, G. Immirzi and B. Ydri, *Fuzzy* CP(2), `hep-th/0103023`.

[21] B. Morariu and A.P. Polychronakos, *Finite noncommutative Chern-Simons with a Wilson line and the quantum hall effect*, *J. High Energy Phys.* **07** (2001) 006 [`hep-th/0106072`].